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Technical Notes

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Temperature–Heat-Flux Integral Relationship in the Half-Space by Fourier Transforms

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Nomenclature

A	=	constant in Holder's condition, Eq. (3e)
C	=	heat capacity, kJ/(kg°C)
$f(x)$	=	initial condition, Eq. (1b) °C
k	=	thermal conductivity, W/(m°C)
q''	=	dimensional heat flux, W/m^2
s	=	dummy time variable, s
T	=	temperature, °C
T_o	=	initial temperature, °C
$\bar{T}_\lambda(t)$	=	Fourier transform (either sine of cosine), °C/m
t	=	time, s
t_o	=	dummy time variable, s
u	=	dummy time variable, s
x	=	spatial variable, m
α	=	thermal diffusivity [$k/(\rho C)$], m^2/s
β	=	constant in Holder's condition, Eq. (3e)
η	=	fixed position, m
λ	=	continuous spectral variable, 1/m
ρ	=	density, kg/m^3

I. Introduction

NOVEL integral relationships have been recently reported [1–7] between heat flux and temperature for constant-property transient heat conduction in the half-space under a variety of conditions. Kulish et al. [1–3] proposed a methodology based on Laplace transforms. Frankel et al. [4–7] developed a unified theoretic methodology applicable to one- and multidimensional geometries [4–7] based on Green's function method and kernel regularization. This Note offers an alternative mathematical treatment using Fourier transforms to produce the identical results previously reported. In addition, this approach requires fewer mathematical tools than Green's function method and will be shown to offer comparable generalization. Analysis involving thin-film heat-flux gauges [8,9], coaxial thermocouples [10], and null-point calorimetry [10] can make use of the in-depth relationships developed within this Note.

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The unsteady constant-property heat-conduction equation in the half-space is given by the linear parabolic equation ([11], page 39)

$$\frac{1}{\alpha}\frac{\partial T}{\partial t}(x,t) = \frac{\partial^2 T}{\partial x^2}(x,t), \qquad (x,t) \geq 0 \tag{1a}$$

subject to the initial condition

$$T(x,0) = f(x), \qquad x \geq 0 \tag{1b}$$

where α is the thermal diffusivity and $f(x)$ is the initial condition. The heat flux is assumed to follow the infinite speed of thermal propagation law given by Fourier's law:

$$q''(x,t) = -k\frac{\partial T}{\partial x}(x,t), \qquad (x,t) \geq 0 \tag{1c}$$

where k is the thermal conductivity. Some heat-flux gauges are based on this constitutive relationship [10,12] and hence contain significant intrinsic assumptions.

Kulish et al. [1–3] and Frankel et al. [4–7] have developed relationships between the local temperature and heat flux that are valid at any location in the half-space under a variety of conditions. Under the assumption $f(x) = T_o = 0$°C, the relationship is [1,3–7]

$$T(x,t) = \sqrt{\frac{1}{\rho C k \pi}}\int_{t_o=0}^{t}\frac{q''(x,t_o)}{\sqrt{t-t_o}}\,\mathrm{d}t_o, \qquad (x,t) \geq 0 \tag{2}$$

Notice that this relationship does not require the specification of a surface ($x = 0$) boundary condition. Equation (2) can be interpreted two ways. First, given the heat flux $q''(x,t)$, then mere numerical integration is required to obtain the local temperature. Second, given the local temperature $T(x,t)$, then the solution of a Volterra integral equation of the first kind is required for determining the local heat flux, $q''(x,t)$. In this case, Eq. (2) is viewed as an Abel integral equation [13–16]. Recognizing that Eq. (2) is an Abel equation for heat flux permits its direct inversion by conventional regularization means. Inverting [4,15], Eq. (2) yields

$$q''(x,t) = \sqrt{\frac{\rho C k}{\pi}}\int_{t_o=0}^{t}\frac{\partial T}{\partial t_o}(x,t_o)\frac{\mathrm{d}t_o}{\sqrt{t-t_o}}, \qquad (x,t) \geq 0 \tag{3a}$$

or, alternatively (with proper interpretation involving Hadamard's finite-part integration [4,17–20]), as

$$q''(x,t) = -\sqrt{\frac{\rho C k}{4\pi}}\int_{t_o=0}^{t}\frac{T(x,t_o)}{(t-t_o)^{3/2}}\,\mathrm{d}t_o, \qquad (x,t) \geq 0 \tag{3b}$$

In fact, Eq. (3a) can be determined from Eq. (2) without resorting to inversion. The energy equation can be expressed as $\rho C(\partial T/\partial t) = -(\partial q''/\partial x)$ and Fourier's law can be expressed, as previously noted, as $q'' = -k(\partial T/\partial x)$. Thus, operating on Eq. (2) with $-k(\partial/\partial x)$, making use of the energy equation to eliminate $\partial q''/\partial x$ in the integrand, and using the definition of heat flux also renders Eq. (3a).

Some additional mathematical and numerical observations are now stated concerning Eqs. (2) and (3a) with regard to specifying $T(x,t)$ or $q''(x,t)$ in the presence of discrete noisy data sets. First, given the heat flux $q''(x,t)$ in Eq. (2), the temperature can be determined by direct numerical calculation. This weakly singular integral poses no numerical difficulties, nor is this calculation ill-posed. Second, if provided discrete noisy temperature data, Eq. (2) now represents an Abel integral equation that is mildly ill-posed [13–16]. That is, the root-mean-square error of the heat flux grows as the sample density increases. Numerical calculations for the heat flux are unstable to input perturbations in the temperature data and hence are ill-posed (with respect to temperature data). The inverted equation presented in Eq. (3a) provides insight into this dilemma. The heat flux $q''(x,t)$ in Eq. (3a) is reconstructed from a weakly singular integral involving the heating/cooling rate $(\partial T/\partial t)(x,t)$. That is, the error in the measurement of temperature is differentiated and thus explains the difficulty. Diffusion naturally damps high-frequency information. Thus, digital filtering should be advocated and implemented to remove the unnecessary high-frequency components in the signal that play havoc on numerical differentiation. Signal-to-noise issues should be considered when designing the digital filter. However, if one could 1) control the high-frequency components in the temperature signal through digital filtering or 2) directly measure the heating/cooling rate $(\partial T/\partial t)(x,t)$, then the ill-posedness could be controlled. It should be finally noted that numerical differentiation of noisy data represents the most basic inverse problem [21,22].

Previous discretizations [8,10] of Eq. (3b) have not been performed in a manner that satisfies Holder's first-derivative condition [4,17] (see [4] for a complete derivation). That is, let us use singularity subtraction such that Holder's first-derivative condition met. Therefore, we rewrite Eq. (3b) as

$$q''(x,t) = -\frac{\sqrt{\rho C k}}{\sqrt{4\pi}}\left\{\int_{t_o=0}^{t}[T(x,t_o)-T(x,t) + (t-t_o)\frac{\partial T}{\partial t}(x,t)]\frac{\mathrm{d}t_o}{(t-t_o)^{\frac{3}{2}}} + T(x,t)\,⨎_{t_o=0}^{t}\frac{\mathrm{d}t_o}{(t-t_o)^{\frac{3}{2}}} - \frac{\partial T}{\partial t}(x,t)\int_{t_o=0}^{t}\frac{\mathrm{d}t_o}{\sqrt{t-t_o}}\right\} \tag{3c}$$

or

$$q''(x,t) = -\frac{\sqrt{\rho C k}}{\sqrt{4\pi}}\left\{\int_{t_o=0}^{t}[T(x,t_o)-T(x,t) + (t-t_o)\frac{\partial T}{\partial t}(x,t)]\frac{\mathrm{d}t_o}{(t-t_o)^{\frac{3}{2}}} - \frac{2T(x,t)}{\sqrt{t}} - 2\sqrt{t}\frac{\partial T}{\partial t}(x,t)\right\} \tag{3d}$$

where Holder's first-derivative condition is explicitly

$$|T(x,t_o) - T(x,t) + (t-t_o)\frac{\partial T}{\partial t}(x,t)| < A|t-t_o|^{\beta+1}, \qquad x \geq 0 \tag{3e}$$

with $|A| < \infty$ and $\beta\epsilon(0,1]$. Imposing Holder's condition makes sense on physical grounds in light of the heat operator [i.e., Eq. (1a), which contains a first time derivative]. Here, Hadamard's integral is denoted by $⨎$ and must be mathematically interpreted accordingly [4,17].

Equation (3d) seems well prepared for a simple numerical quadrature if provided both temperature and heating/cooling rate data. It is interesting to note that this formulation makes use of the entire temperature data set but only requires the last heating/cooling rate value. Again, this formulation clearly indicates that Eq. (3b) is ill-posed because differentiated data are still required. Any simple numerical quadrature rule is sufficient to yield accurate results if control on the heating/cooling rate term is maintained as the sample density is increased.

Frankel et al. [4–7] have developed a theoretic formalism based on a full-space Green's function to encompass a multidimensional half-space. This insight can now be brought to a less sophisticated formalism using Fourier transforms. Hence, this Note develops a Fourier transform [23,24] approach to arrive at Eqs. (2) and (3).

II. Derivation of Heat-Flux–Temperature Integral Relationship by Fourier Transforms

Fourier sine and cosine transforms can be applied to half-space geometries [23,24] in a natural manner to remove the spatial variable while retaining the temporal variable. Time as well as space both involve half-space geometries. However, it has been demonstrated by Frankel et al. [4–7], through the application of the full-space Green's function associated with the heat operator, that a high degree of mathematical flexibility is generated by transforming space. It is the purpose of the present Note to use Fourier cosine and sine transforms for obtaining the desired integral relationship in one-dimensional half-space geometries. In a future paper, a two-dimensional extension will be presented. In general, the extension from one-dimensional to two-dimensional studies often requires a substantive observation. The extension from a two-dimensional to a three-dimensional half-space study then becomes relatively straightforward.

Fourier cosine transform: Let the symmetric transform pair be given as ([23], page 17)

Cosine transform:

$$\bar{T}_\lambda(t) = \sqrt{\frac{2}{\pi}}\int_{x=0}^{\infty} T(x,t)\cos(\lambda x)\,\mathrm{d}x, \qquad (\lambda,t) \geq 0 \tag{4a}$$

Inversion:

$$T(x,t) = \sqrt{\frac{2}{\pi}}\int_{\lambda=0}^{\infty} \bar{T}_\lambda(t)\cos(\lambda x)\,\mathrm{d}\lambda, \qquad (x,t) \geq 0 \tag{4b}$$

Infinite physical domains produce a continuous spectral variable λ (continuous eigenvalues), unlike finite physical domains, which involve the discrete spectral parameter normally denoted as λ_m (the mth discrete eigenvalue).

To begin, we operate on the heat equation given in Eq. (1a) with

$$\sqrt{\frac{2}{\pi}}\int_{x=0}^{\infty}\cos(\lambda x)\,\mathrm{d}x$$

to obtain

$$\frac{1}{\alpha}\sqrt{\frac{2}{\pi}}\int_{x=0}^{\infty}\frac{\partial T}{\partial t}(x,t)\cos(\lambda x)\,\mathrm{d}x = \sqrt{\frac{2}{\pi}}\int_{x=0}^{\infty}\frac{\partial^2 T}{\partial x^2}(x,t)\cos(\lambda x)\,\mathrm{d}x, \qquad (\lambda,t) \geq 0 \tag{5a}$$

Interchanging the orders of integration and time differentiation on the left-hand side, integrating the right-hand side by parts twice, and incorporating the boundary conditions at zero and infinity $((\partial T/\partial x)(\infty,t) = 0)$, we formally obtain

$$\frac{1}{\alpha}\frac{\mathrm{d}}{\mathrm{d}t}\sqrt{\frac{2}{\pi}}\int_{x=0}^{\infty}T(x,t)\cos(\lambda x)\,\mathrm{d}x = -\sqrt{\frac{2}{\pi}}\frac{\partial T}{\partial x}(0,t) - \lambda^2\sqrt{\frac{2}{\pi}}\int_{x=0}^{\infty}T(x,t)\cos(\lambda x)\,\mathrm{d}x, \qquad (\lambda,t) \geq 0 \tag{5b}$$

The definition of the cosine transform shown in Eq. (4a) is identifiable in Eq. (5b); therefore, Eq. (5b) can be expressed by the first-order initial-value problem in the transformed variable as

$$\frac{\mathrm{d}\bar{T}_\lambda}{\mathrm{d}t}(t) + \alpha\lambda^2\bar{T}_\lambda(t) = \sqrt{\frac{2}{\pi}}\frac{\alpha}{k}q''(0,t), \qquad (\lambda,t) \geq 0 \tag{5c}$$

where the definition of Fourier's heat flux [Eq. (1c)] has been used. The solution to Eq. (5c) is readily attainable by an integrating factor. The solution for the Fourier cosine transform $\bar{T}_\lambda(t)$ in terms of unknown surface heat flux becomes

$$\bar{T}_\lambda(t) = \sqrt{\frac{2}{\pi}}\frac{\alpha}{k}\int_{u=0}^{t} e^{-\alpha\lambda^2(t-u)}q''(0,u)\,\mathrm{d}u, \qquad (\lambda,t) \geq 0 \tag{6a}$$

where $\bar{T}_\lambda(0) = 0$ from Eq. (4b) because $T(x,0) = f(x) = T_o = 0°\text{C}$. At this juncture, the surface heat flux is not specified. Next, we substitute Eq. (6a) into the inversion formula displayed in Eq. (4b) and upon interchanging orders of integration, we arrive at

$$T(x,t) = \frac{1}{\sqrt{\rho C k \pi}} \int_{u=0}^{t} q''(0,u) \frac{e^{-\frac{x^2}{4\alpha(t-u)}}}{\sqrt{t-u}} \, \mathrm{d}u, \qquad (x,t) \geq 0 \quad \text{(6b)}$$

where

$$\int_{\lambda=0}^{\infty} e^{-\alpha\lambda^2(t-u)} \cos(\lambda x) \, \mathrm{d}\lambda = \frac{\sqrt{\pi}}{2\sqrt{\alpha}} \frac{e^{-\frac{x^2}{4\alpha(t-u)}}}{\sqrt{t-u}}, \qquad (\lambda,t) \geq 0 \quad \text{(6c)}$$

with $\alpha = k/(\rho C)$. Hence, if given $q''(0,t)$ then the temperature distribution (solution) is known for all $(x,t) \geq 0$ from Eq. (6b). Next, the heat-flux distribution $q''(x,t)$ for all $(x,t) \geq 0$ can be obtained by Fourier's law. That is, we operate on Eq. (6b) with $-k(\partial/\partial x)$ and simplify to obtain

$$q''(x,t) = -k \frac{\partial T}{\partial x}(x,t) = \frac{x}{2\sqrt{\alpha\pi}} \int_{u=0}^{t} q''(0,u) \frac{e^{-\frac{x^2}{4\alpha(t-u)}}}{(t-u)^{\frac{3}{2}}} \, \mathrm{d}u, \qquad (x,t) > 0 \quad \text{(6d)}$$

hence, if given $q''(0,t)$, then Eq. (6d) is the corresponding flux distribution (solution) for $x > 0$.

Equations (6b) and (6d) are useful expressions for determining the temperature and heat-flux distributions based on a specified surface heat flux, $q''(0,t)$. However, it is our intent to develop an integral relationship between temperature and heat flux valid at location $x \geq 0$ without knowledge of $q''(0,t)$. Hence, we must analytically eliminate this function or integral involving $q''(0,t)$. To develop this relationship, we observe that the kernel displayed in Eq. (6b) contains a familiar form (i.e., a portion of an Abel kernel). This suggests performing a "regularization" operation, as associated with singular kernels. That is, let $t \to s$ and operate on the resultant with

$$\int_{s=0}^{t} \frac{\mathrm{d}s}{\sqrt{t-s}}$$

to obtain

$$\int_{s=0}^{t} T(x,s) \frac{\mathrm{d}s}{\sqrt{t-s}} = \frac{1}{\sqrt{\rho C k \pi}} \int_{s=0}^{t} \frac{\mathrm{d}s}{\sqrt{t-s}} \int_{u=0}^{s} q''(0,u) \frac{e^{-\frac{x^2}{4\alpha(s-u)}}}{\sqrt{s-u}} \, \mathrm{d}u, \qquad (x,t) \geq 0 \quad \text{(7a)}$$

The double integral on the right-hand side can be reduced to a single integral because the integration region is defined by a triangle. Carefully interchanging orders of integration produces

$$\int_{s=0}^{t} T(x,s) \frac{\mathrm{d}s}{\sqrt{t-s}} = \frac{1}{\sqrt{\rho C k \pi}} \int_{u=0}^{t} q''(0,u) \int_{s=u}^{t} \frac{e^{-\frac{x^2}{4\alpha(s-u)}}}{\sqrt{t-s}\sqrt{s-u}} \, \mathrm{d}s \, \mathrm{d}u, \qquad (x,t) \geq 0 \quad \text{(7b)}$$

or upon analytic integration and letting $s \to u$ on the left-hand side for cosmetics, we arrive at

$$\int_{u=0}^{t} T(x,u) \frac{\mathrm{d}u}{\sqrt{t-u}} = \sqrt{\frac{\pi}{\rho C k}} \int_{u=0}^{t} q''(0,u) \, erfc \sqrt{\frac{x^2}{4\alpha(t-u)}} \, \mathrm{d}u, \qquad (x,t) \geq 0 \quad \text{(7c)}$$

with

$$\int_{s=u}^{t} \frac{e^{-\frac{x^2}{4\alpha(s-u)}}}{\sqrt{t-s}\sqrt{s-u}} \, \mathrm{d}s = \pi \, erfc \sqrt{\frac{x^2}{4\alpha(t-u)}} \quad \text{(7d)}$$

where $erfc(z)$ is the complementary error function ([25], page 297) with real argument z.

At this junction, we have two options to arrive at a relationship that can be used in conjunction with Eq. (6d). The first (and equivalent) option involves integrating the left-hand side of Eq. (7c) by parts to produce

$$2\int_{u=0}^{t} \frac{\partial T}{\partial u}(x,u)\sqrt{t-u} \, \mathrm{d}u = \sqrt{\frac{\pi}{\rho C k}} \int_{u=0}^{t} q''(0,u) \, erfc \sqrt{\frac{x^2}{4\alpha(t-u)}} \, \mathrm{d}u, \qquad (x,t) \geq 0 \quad \text{(8a)}$$

because $T(x,0) = f(x) = T_o = 0°\text{C}$. Next, we operate on Eq. (8a) with $\partial/\partial t$, and upon using Leibnitz's rule (for $x > 0$), we obtain

$$\int_{u=0}^{t} \frac{\partial T}{\partial u}(x,u) \frac{\mathrm{d}u}{\sqrt{t-u}} = \sqrt{\frac{\pi}{\rho C k}} \int_{u=0}^{t} q''(0,u) \frac{\partial}{\partial t}\left(erfc \sqrt{\frac{x^2}{4\alpha(t-u)}} \right) \mathrm{d}u, \qquad (x,t) > 0 \quad \text{(8b)}$$

or

$$\int_{u=0}^{t} \frac{\partial T}{\partial u}(x,u) \frac{\mathrm{d}u}{\sqrt{t-u}} = \frac{x}{2k} \int_{u=0}^{t} q''(0,u) \frac{e^{-\frac{x^2}{4\alpha(t-u)}}}{(t-u)^{\frac{3}{2}}} \, \mathrm{d}u, \qquad (x,t) > 0 \quad \text{(8c)}$$

because

$$\frac{\partial}{\partial t}\left(erfc \sqrt{\frac{x^2}{4\alpha(t-u)}} \right) = \frac{x}{2\sqrt{\alpha\pi}} \frac{e^{-\frac{x^2}{4\alpha(t-u)}}}{(t-u)^{\frac{3}{2}}} \quad \text{(8d)}$$

The right-hand side of Eq. (8c) can be analytically identified with the aid of Eq. (6d). If Eq. (6d) is multiplied by $\sqrt{\alpha\pi}/k$, then the right-hand side of Eq. (6d) matches the right-hand side of Eq. (8c). Thus, Eq. (8c) becomes

$$\int_{u=0}^{t} \frac{\partial T}{\partial u}(x,u) \frac{\mathrm{d}u}{\sqrt{t-u}} = \frac{\sqrt{\alpha\pi}}{k} q''(x,t), \qquad (x,t) > 0 \quad \text{(8e)}$$

or, finally, we obtain the important result (which is also valid at $x = 0$):

$$q''(x,t) = \sqrt{\frac{\rho C k}{\pi}} \int_{u=0}^{t} \frac{\partial T}{\partial u}(x,u) \frac{\mathrm{d}u}{\sqrt{t-u}}, \qquad (x,t) \geq 0 \quad \text{(8f)}$$

The second option involves immediately operating on Eq. (7c) with $\partial^2/\partial x^2$ to obtain

$$\int_{u=0}^{t} \frac{\partial^2 T}{\partial x^2}(x,u) \frac{\mathrm{d}u}{\sqrt{t-u}} = \sqrt{\frac{\pi}{\rho C k}} \int_{u=0}^{t} q''(0,u) \frac{\partial^2}{\partial x^2}\left(erfc \sqrt{\frac{x^2}{4\alpha(t-u)}} \right) \mathrm{d}u, \qquad (x,t) \geq 0 \quad \text{(9a)}$$

or

$$\int_{u=0}^{t} \frac{\partial T}{\partial u}(x,u) \frac{\mathrm{d}u}{\sqrt{t-u}} = \frac{x}{2k} \int_{u=0}^{t} q''(0,u) \frac{e^{-\frac{x^2}{4\alpha(t-u)}}}{(t-u)^{\frac{3}{2}}} \, \mathrm{d}u, \qquad (x,t) \geq 0 \quad \text{(9b)}$$

because

$$\frac{\partial^2}{\partial x^2}\left(erfc\sqrt{\frac{x^2}{4\alpha(t-u)}}\right) = \frac{x}{2\sqrt{\pi}}\frac{e^{-\frac{x^2}{4\alpha(t-u)}}}{(\alpha(t-u))^{\frac{3}{2}}} \tag{9c}$$

and where

$$\frac{1}{\alpha}\frac{\partial T}{\partial u}(x,u) = \frac{\partial^2 T}{\partial x^2}(x,u)$$

Equation (9b) is identical to Eq. (8c) and hence reduces to Eq. (8f) following a similar logic.

Equation (8f) is the desired relationship between the local heat flux and temperature (or explicitly heating rate, $\partial T/\partial t$). Stability is indicated by the heating rate [4]. Equation (8f) can equivalently be expressed in terms of temperature. Integrating the right-hand side in Eq. (8f) by parts (where integration is based on Hadamard's finite part [4,17–20]) and incorporating the initial condition $T(x,0) = f(x) = T_o = 0°\mathrm{C}$ produces

$$q''(x,t) = -\frac{k}{2\sqrt{\alpha\pi}}\int_{u=0}^{t}\frac{T(x,u)}{(t-u)^{\frac{3}{2}}}\,\mathrm{d}u \qquad (x,t)\geq 0 \tag{9d}$$

which is identical to Eq. (3b) because $\alpha = k/(\rho C)$. Equations (8f) and (9d) represents the inversion for the temperature–heat-flux relationship when provided $T(x,t)$ in some manner. In practice, Eq. (8f) is more useful than Eq. (9d) and clearly suggests a new sensor strategy for measuring heat flux in transient problems. That is, suppose a temperature sensor is located at $x = \eta > 0$ and data are collected at a fixed sample rate. If raw temperature data are directly used in Eq. (9d), then it has been previously noted that this problem is ill-posed. That is, the accuracy in the approximation for heat flux decreases as the sample rate increases. However, it is well known that diffusion damps out high frequencies. With this knowledge, the application of a low-pass digital filter is highly acceptable once a cutoff frequency is estimated. The chosen filter function should be designed with the time-derivative resultant in mind, as indicated in Eq. (8f). The filtered temperature data can then be used in Eq. (8f) to produce an accurate approximation to the local heat flux $q''(\eta,t)$ based on a simple numerical quadrature rule. Thus, the transient heat flux can be determined from a single thermocouple in the half-space.

Fourier sine transform: Let the symmetric transform pair be given as ([23], page 18)

Sine transform:

$$\bar{T}_\lambda(t) = \sqrt{\frac{2}{\pi}}\int_{x=0}^{\infty} T(x,t)\sin(\lambda x)\,\mathrm{d}x, \qquad (\lambda,t)\geq 0 \tag{10a}$$

Inversion:

$$T(x,t) = \sqrt{\frac{2}{\pi}}\int_{\lambda=0}^{\infty}\bar{T}_\lambda(t)\sin(\lambda x)\,\mathrm{d}\lambda, \qquad (x,t)\geq 0 \tag{10b}$$

To begin, we operate on the heat equation given in Eq. (1a) with

$$\sqrt{\frac{2}{\pi}}\int_{x=0}^{\infty}\sin(\lambda x)\,\mathrm{d}x$$

to obtain

$$\frac{1}{\alpha}\sqrt{\frac{2}{\pi}}\int_{x=0}^{\infty}\frac{\partial T}{\partial t}(x,t)\sin(\lambda x)\,\mathrm{d}x = \sqrt{\frac{2}{\pi}}\int_{x=0}^{\infty}\frac{\partial^2 T}{\partial x^2}(x,t)\sin(\lambda x)\,\mathrm{d}x, \qquad (\lambda,t)\geq 0 \tag{11a}$$

Interchanging the orders of integration and time differentiation on the left-hand side, integrating the right-hand side by parts twice, and incorporating the boundary conditions at zero and infinity ($T(\infty,t) = T_o = 0°\mathrm{C}$), we formally obtain

$$\frac{1}{\alpha}\frac{d}{dt}\sqrt{\frac{2}{\pi}}\int_{x=0}^{\infty}T(x,t)\sin(\lambda x)\,\mathrm{d}x = \sqrt{\frac{2}{\pi}}\lambda T(0,t) - \lambda^2\sqrt{\frac{2}{\pi}}\int_{x=0}^{\infty}T(x,t)\sin(\lambda x)\,\mathrm{d}x, \qquad (\lambda,t)\geq 0 \tag{11b}$$

The definition of the sine transform shown in Eq. (10a) is identifiable in Eq. (11b); therefore, Eq. (11b) can be expressed by the first-order initial-value problem in the transformed variable as

$$\frac{\mathrm{d}\bar{T}_\lambda}{\mathrm{d}t}(t) + \alpha\lambda^2\bar{T}_\lambda(t) = \sqrt{\frac{2}{\pi}}\lambda\alpha T(0,t), \qquad (\lambda,t)\geq 0 \tag{11c}$$

The solution to Eq. (11c) is readily attainable by an integrating factor. The solution for the Fourier sine transform $\bar{T}_\lambda(t)$ in terms of unknown surface temperature becomes

$$\bar{T}_\lambda(t) = \lambda\alpha\sqrt{\frac{2}{\pi}}\int_{u=0}^{t}T(0,u)e^{-\alpha\lambda^2(t-u)}\,\mathrm{d}u, \qquad (\lambda,t)\geq 0 \tag{11d}$$

where $\bar{T}_\lambda(0) = 0$ from Eq. (10b) because $T(x,0) = f(x) = T_o = 0°\mathrm{C}$. Next, we substitute this result for $\bar{T}_\lambda(t)$ into the inversion formula displayed in Eq. (10b) and interchange orders of integration to get

$$T(x,t) = \frac{2\alpha}{\pi}\int_{u=0}^{t}T(0,u)\int_{\lambda=0}^{\infty}\lambda e^{-\alpha\lambda^2(t-u)}\sin(\lambda x)\,\mathrm{d}\lambda\,\mathrm{d}u, \qquad (x,t)\geq 0 \tag{12a}$$

The inside integral can be integrated analytically reducing Eq. (12a) to

$$T(x,t) = \frac{x}{2\sqrt{\alpha\pi}}\int_{u=0}^{t}T(0,u)\frac{e^{-\frac{x^2}{4\alpha(t-u)}}}{(t-u)^{\frac{3}{2}}}\,\mathrm{d}u, \qquad (x,t)\geq 0 \tag{12b}$$

where we made use of

$$\int_{\lambda=0}^{\infty}\lambda e^{-\alpha\lambda^2(t-u)}\sin(\lambda x)\,\mathrm{d}\lambda = \frac{\sqrt{\pi}}{4}e^{-\frac{x^2}{4\alpha(t-u)}}\sqrt{\frac{x^2}{(\alpha(t-u))^3}} \tag{12c}$$

Equation (12b) represents the solution for the temperature distribution if given $T(0,t)$. The heat-flux $q''(x,t)$ distribution is developed by operating on Eq. (12b) with $-k(\partial/\partial x)$. Thus, we obtain

$$q''(x,t) = -k\frac{\partial T}{\partial x}(x,t) = -\frac{k}{2\sqrt{\alpha\pi}}\int_{u=0}^{t}T(0,u)\frac{\partial}{\partial x}\left(\frac{xe^{-\frac{x^2}{4\alpha(t-u)}}}{(t-u)^{\frac{3}{2}}}\right)\mathrm{d}u, \qquad (x,t)\geq 0 \tag{13a}$$

or upon some simplifications, we find

$$q''(x,t) = -\frac{k}{2\sqrt{\alpha\pi}}\int_{u=0}^{t}\frac{T(0,u)}{(t-u)^{\frac{3}{2}}}e^{-\frac{x^2}{4\alpha(t-u)}}\left(1-\frac{x^2}{2\alpha(t-u)}\right)\mathrm{d}u, \qquad (x,t)\geq 0 \tag{13b}$$

Equation (13b) represents the heat-flux distribution (solution) if provided $T(0,t)$. However, it is our intent to develop a relationship between temperature and heat flux that does not require knowledge of the surface condition. To do this, let us regularize Eq. (12b) as before. To begin the process, let $t \to s$ in Eq. (12b) and then operate on the result with

$$\int_{s=0}^{t}\frac{\mathrm{d}s}{\sqrt{t-s}}$$

to get

$$\int_{s=0}^{t}\frac{T(x,s)}{\sqrt{t-s}}\,\mathrm{d}s = \frac{x}{2\sqrt{\alpha\pi}}\int_{s=0}^{t}\frac{\mathrm{d}s}{\sqrt{t-s}}\int_{u=0}^{s}T(0,u)\frac{e^{-\frac{x^2}{4\alpha(s-u)}}}{(s-u)^{\frac{3}{2}}}\,\mathrm{d}u, \qquad (x,t)\geq 0 \tag{13c}$$

Interchanging orders of integration on the triangle on the right-hand side and letting $s\to u$ on the left-hand side, we obtain

$$\int_{u=0}^{t}\frac{T(x,u)}{\sqrt{t-u}}\,\mathrm{d}u = \int_{u=0}^{t}T(0,u)\frac{e^{-\frac{x^2}{4\alpha(t-u)}}}{\sqrt{t-u}}\,\mathrm{d}u, \qquad (x,t)\geq 0 \tag{13d}$$

where

$$\int_{s=u}^{t}\frac{e^{-\frac{x^2}{4\alpha(s-u)}}}{\sqrt{t-s}(s-u)^{\frac{3}{2}}}\,\mathrm{d}s = \sqrt{\frac{4\alpha\pi}{x^2}}\frac{e^{-\frac{x^2}{4\alpha(t-u)}}}{\sqrt{t-u}} \tag{13e}$$

Following the second option described previously, we next operate on Eq. (13d) with $\partial^2/\partial x^2$ to get

$$\int_{u=0}^{t}\frac{\partial^2 T}{\partial x^2}(x,u)\frac{\mathrm{d}u}{\sqrt{t-u}} = \int_{u=0}^{t}T(0,u)\frac{\partial^2}{\partial x^2}\left(\frac{e^{-\frac{x^2}{4\alpha(t-u)}}}{\sqrt{t-u}}\right)\mathrm{d}u, \qquad (x,t)\geq 0 \tag{14a}$$

Making use of the heat equation

$$\frac{1}{\alpha}\frac{\partial T}{\partial u}(x,u) = \frac{\partial^2 T}{\partial x^2}(x,u)$$

and performing the indicated differentiations on the right-hand side yields

$$\int_{u=0}^{t}\frac{\partial T}{\partial u}(x,u)\frac{\mathrm{d}u}{\sqrt{t-u}} = -\frac{1}{2}\int_{u=0}^{t}T(0,u)\frac{e^{-\frac{x^2}{4\alpha(t-u)}}}{(t-u)^{\frac{3}{2}}}\left(1-\frac{x^2}{2\alpha(t-u)}\right)\mathrm{d}u, \qquad (x,t)\geq 0 \tag{14b}$$

Again, we observe that the right-hand side of Eq. (14b) is similar to that displayed in Eq. (13b). Equation (13b) can be expressed in the alternative form

$$\sqrt{\frac{\alpha\pi}{k}}q''(x,t) = -\frac{1}{2}\int_{u=0}^{t}\frac{T(0,u)}{(t-u)^{\frac{3}{2}}}e^{-\frac{x^2}{4\alpha(t-u)}}\left(1-\frac{x^2}{2\alpha(t-u)}\right)\mathrm{d}u, \qquad (x,t)\geq 0 \tag{14c}$$

and hence Eq. (14b) becomes

$$q''(x,t) = \sqrt{\frac{\rho Ck}{\pi}}\int_{u=0}^{t}\frac{\partial T}{\partial u}(x,u)\frac{\mathrm{d}u}{\sqrt{t-u}}, \qquad (x,t)\geq 0 \tag{14d}$$

which is identical to Eq. (8f).

III. Conclusions

An integral relationship between the heat flux and temperature (or heating rate) leads to a useful transient means for determining the local heat flux in a half-space. This relationship warrants significant consideration for future applications. Generalization of the approach to two-dimensional systems [7] has been recently reported. In transient situations, Eq. (8f) is highly useful for estimating the heat flux for $x\geq 0$ and offers significant advantages over conventional heat-flux gauges if in-depth measurements are required. The local heat flux is obtained from either filtering temperature data or directly from a heating/cooling rate sensor. Finally, a recently developed voltage-rate interface [26] has been demonstrated to yield remarkably accurate heating/cooling rates.

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